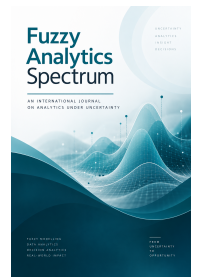


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Alpha-Cut Fuzzy Tree Parameters: Width, Length, Depth, Breadth, and Connections to Fuzzy Decision Trees

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ABSTRACT

This paper develops an alpha-cut framework for four classical graph parameters on finite undirected fuzzy graphs: tree-width, tree-length, tree-depth, and tree-breadth. Each parameter is evaluated on the crisp graph obtained at a positive membership threshold, and its profile is integrated over $[0, 1]$. We show that these profiles are bounded step functions and obtain exact finite-sum formulas for the integrated parameters. We establish threshold monotonicity for width and depth, crisp reduction, and inequalities connecting the integrated length and breadth parameters. A strengthened width–depth inequality explicitly accounts for the maximum vertex-membership degree. We also examine fuzzy decision trees under a normalized product-routing model. For a graph induced by test-node activations and ancestor relations, we derive exact expressions for integrated width and depth and establish bounds involving soft path length. Compatible feature-interaction hierarchies provide further structural bounds. These findings distinguish alpha-cut aggregation from fuzzy-bag approaches and characterize its relationship to selected decision-tree structures without making claims about predictive performance.

1. Introduction

1.1 Fuzzy Graphs and Alpha-Cut Representations

Classical graphs represent adjacency as a binary relation, whereas many relational descriptions also involve graded or uncertain information. Fuzzy sets provide a standard mathematical framework for graded membership: a membership function assigns each element a value in $[0, 1]$ [1]. Related models include intuitionistic fuzzy sets [2, 3] and neutrosophic sets [4, 5]. Further extensions and generalizations, including Plithogenic Sets [6, 7], have also been developed.

In graph theory, fuzzy graphs assign membership degrees to vertices and edges, subject to compatibility conditions in the model considered here [8, 9, 10]. Other uncertainty frameworks have motivated related graph structures, including intuitionistic fuzzy and neutrosophic graphs [3, 11, 12, 13, 14]. The present study focuses on finite undirected fuzzy graphs with vertex membership σ and edge membership μ , where $\mu(uv) \leq \min\{\sigma(u), \sigma(v)\}$. Related concepts, including Fuzzy HyperGraphs [15, 16] and Fuzzy SuperHyperGraphs [17], have also been studied.

For each $\alpha \in (0, 1]$, the α -cut retains exactly the vertices and edges whose membership values are at least α . These cuts form a nested family of ordinary graphs. Unlike the support graph, which records

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only positive membership, the family of α -cuts retains information about where the graph changes as the threshold varies. Because the underlying fuzzy graph is finite, only finitely many distinct positive membership levels need to be considered when integrating a graph-parameter profile.

1.2 Tree Parameters and Fuzzy Decision Trees

Tree-decompositions describe a graph through overlapping vertex bags indexed by a tree. Tree-Width minimizes the largest bag size minus one [18, 19, 20], whereas Path-Width imposes a path structure on the decomposition [21, 22]. Tree-Length and Tree-Breadth measure metric properties of decomposition bags using graph distances and covering radii, respectively [23, 24]. Tree-Depth is defined through minimum-height rooted forests whose closures contain the graph [25]. Other decomposition or layout parameters, such as Hypertree-Width, Branch-Width, Rank-Width, and MIM-Width, study different aspects of graph structure [26, 27, 28, 29, 30].

Existing fuzzy-bag approaches place membership degrees directly into tree-decomposition bags and the associated width measures [31, 32]. By contrast, we retain the classical parameters, evaluate them on every positive α -cut, and integrate their profiles. This gives one threshold-based treatment of Tree-Width, Tree-Length, Tree-Depth, and Tree-Breadth. We distinguish the monotonicity results for width and depth from those for the metric parameters: the distances used by Tree-Length and Tree-Breadth may change when the threshold changes.

Decision Trees are hierarchical models that recursively partition data using feature-based conditions, supporting classification or regression through successive decisions at internal nodes [33, 34]. Fuzzy Decision Trees provide another motivation for considering structural and hierarchical quantities under graded membership. Such models use fuzzy branch memberships for classification or inference [35, 36]. Related concepts, including Neutrosophic Decision Trees, have also been investigated [37, 38]. Their operational depth, however, is not identical to the graph-theoretic Tree-Depth parameter. To make this distinction precise, we construct a fuzzy graph whose vertices are occurrences of internal test nodes and whose memberships are determined by normalized product-routing activations. We also consider rooted forests compatible with the support graph of a fuzzy feature-interaction structure. These constructions yield mathematical relations between tree parameters and selected decision-tree hierarchies, without presupposing bounds on classification accuracy or computational cost.

The two approaches to fuzzy decomposition parameters are compared in Table 1.

Table 1
Comparison of fuzzy-bag and alpha-cut viewpoints for tree parameters.

Aspect	Fuzzy-bag viewpoint	Alpha-cut viewpoint
Membership treatment	Membership values are incorporated directly into fuzzy bags or graded bag-membership conditions.	Membership values determine the family of crisp threshold graphs $\{G_\alpha\}_{\alpha \in (0,1]}$.
Underlying decomposition	A fuzzy tree-decomposition is considered directly.	An ordinary classical parameter is evaluated separately on each G_α .
Threshold profile	A threshold profile is not the defining object of the formulation.	The function $\alpha \mapsto p(G_\alpha)$ is an explicit primary object.
Scalar summary	Depends on the particular fuzzy-bag formulation.	The threshold profile is integrated over $[0, 1]$ to obtain a scalar summary.

1.3 Our Contributions

This paper establishes the following results for finite undirected fuzzy graphs and the specified decision-tree constructions:

- We introduce alpha-cut profiles and their integrated counterparts for Tree-Width, Tree-Length, Tree-Depth, and Tree-Breadth.

- We show that all four profiles are bounded step functions and derive exact finite-sum expressions over the distinct membership levels (Theorems 4.1 and 4.2).
- We establish threshold monotonicity for width and depth, inequalities relating Tree-Length to Tree-Breadth, and crisp reduction (Theorems 4.4 and 4.5).
- We strengthen the integrated width–depth comparison by accounting for the maximum vertex-membership degree (Theorem 5.10).
- For normalized product-routing Fuzzy Decision Trees, we define an induced fuzzy graph and obtain exact integrated width and depth formulas, together with soft path-length bounds (Theorem 5.3 and Corollary 5.4).
- We derive additional structural bounds from feature-interaction-compatible rooted forests (Theorem 5.8) and illustrate the effect of membership levels on the integrated parameters.

2. Preliminaries

Throughout the paper, all graphs are finite, undirected, and loopless unless stated otherwise.

2.1 Fuzzy Graphs, Support Graphs, and Alpha-Cuts

A fuzzy graph assigns membership degrees in $[0, 1]$ to vertices and edges, with each edge degree bounded by the membership degrees of its endpoints. The support graph retains precisely those vertices and edges with positive membership degrees. An α -cut graph retains those vertices and edges whose membership degrees are at least a specified positive threshold.

Definition 2.1 (Finite Undirected Fuzzy Graph). [8, 10] A *finite undirected fuzzy graph* is a triple

$$\tilde{G} = (V, \sigma, \mu),$$

where V is a finite set,

$$\sigma : V \longrightarrow [0, 1]$$

is the vertex-membership function, and

$$\mu : \binom{V}{2} \longrightarrow [0, 1]$$

is the edge-membership function satisfying

$$\mu(uv) \leq \min\{\sigma(u), \sigma(v)\}$$

for every $uv \in \binom{V}{2}$.

Definition 2.2 (Support Graph and Alpha-Cut Graph). Let

$$\tilde{G} = (V, \sigma, \mu)$$

be a finite undirected fuzzy graph. Define

$$V^+ = \{v \in V : \sigma(v) > 0\}$$

and

$$E^+ = \left\{ uv \in \binom{V^+}{2} : \mu(uv) > 0 \right\}.$$

The graph

$$G^+ = (V^+, E^+)$$

is called the *support graph* of $\tilde{G}$.

For each $\alpha \in (0, 1]$, define

$$V_\alpha = \{v \in V : \sigma(v) \geq \alpha\}$$

and

$$E_\alpha = \left\{ uv \in \binom{V_\alpha}{2} : \mu(uv) \geq \alpha \right\}.$$

The graph

$$G_\alpha = (V_\alpha, E_\alpha)$$

is called the α -cut graph of $\tilde{G}$.

Remark 2.3. The compatibility condition

$$\mu(uv) \leq \min\{\sigma(u), \sigma(v)\}$$

ensures that $\mu(uv) \geq \alpha$ implies $u, v \in V_\alpha$. Hence every α -cut graph is well-defined. Moreover, if

$$0 < \alpha \leq \beta \leq 1,$$

then

$$G_\beta \subseteq G_\alpha.$$

2.2 Classical Tree Parameters

We recall the classical tree parameters used in the alpha-cut constructions. A tree-decomposition represents a graph by vertex bags arranged in a tree, satisfying vertex coverage, edge coverage, and connectedness conditions [19, 20]. Tree-width measures how tree-like a graph is by minimizing the maximum bag size minus one over all tree-decompositions [39, 40, 41].

Definition 2.4 (Tree-Decomposition and Tree-Width). Let H be a finite graph. A *tree-decomposition* of H is a pair

$$\mathcal{T} = (T, \{X_t\}_{t \in V(T)}),$$

where T is a tree and each $X_t \subseteq V(H)$, such that:

1. for every $v \in V(H)$, the set

$$\{t \in V(T) : v \in X_t\}$$

is nonempty and induces a connected subtree of T ;

2. for every $uv \in E(H)$, there exists $t \in V(T)$ such that

$$\{u, v\} \subseteq X_t.$$

The width of $\mathcal{T}$ is

$$\text{width}(\mathcal{T}) = \max_{t \in V(T)} |X_t| - 1,$$

and the *Tree-Width* of H is

$$\text{tw}(H) = \min_{\mathcal{T}} \text{width}(\mathcal{T}).$$

We set $\text{tw}(\emptyset) = 0$ for the alpha-cut framework.

For a possibly disconnected finite graph H , let d_H denote the extended shortest-path distance, with

$$d_H(u, v) = \infty$$

when u and v lie in different connected components. For notational convenience, we set

$$\text{diam}_H(\emptyset) = 0.$$

For a nonempty graph, this convention is harmless because empty bags can be deleted from a tree-decomposition without affecting the decomposition conditions or any minimum considered below.

Definition 2.5 (Tree-Length). [23] For a nonempty set $X \subseteq V(H)$, define

$$\text{diam}_H(X) = \max_{u, v \in X} d_H(u, v).$$

For a tree-decomposition

$$\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$$

of H , define

$$\text{len}_H(\mathcal{T}) = \max_{t \in V(T)} \text{diam}_H(X_t).$$

The *Tree-Length* of H is

$$\text{tl}(H) = \min_{\mathcal{T}} \text{len}_H(\mathcal{T}).$$

We set $\text{tl}(\emptyset) = 0$.

Definition 2.6 (Tree-Depth). [25] Let F be a rooted forest. Its closure $\text{clos}(F)$ is the graph on $V(F)$ in which two distinct vertices are adjacent whenever one is an ancestor of the other in F . Let $h(F)$ be the maximum number of vertices on a root-to-leaf path in F . The *Tree-Depth* of a nonempty graph H is

$$\text{td}(H) = \min \{h(F) : H \subseteq \text{clos}(F), V(F) = V(H)\}.$$

We set $\text{td}(\emptyset) = 0$.

Definition 2.7 (Tree-Breadth). [24] We set $\text{rad}_H(\emptyset) = 0$. For a nonempty set $X \subseteq V(H)$, define its radius in H by

$$\text{rad}_H(X) = \min_{c \in V(H)} \max_{v \in X} d_H(c, v).$$

Equivalently, $\text{rad}_H(X)$ is the least $r \geq 0$ for which there exists $c \in V(H)$ satisfying

$$X \subseteq B_H(c, r), \quad B_H(c, r) = \{v \in V(H) : d_H(c, v) \leq r\}.$$

For a tree-decomposition

$$\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$$

of H , define

$$\text{breadth}_H(\mathcal{T}) = \max_{t \in V(T)} \text{rad}_H(X_t).$$

The *Tree-Breadth* of H is

$$\text{tb}(H) = \min_{\mathcal{T}} \text{breadth}_H(\mathcal{T}).$$

We set $\text{tb}(\emptyset) = 0$.

Remark 2.8. The use of the extended distance causes no difficulty for disconnected finite graphs. A tree-decomposition may be formed componentwise and the corresponding decomposition trees may then be joined by arbitrary tree edges without changing any bag. Thus Tree-Length and Tree-Breadth remain finite.

3. Alpha-Cut Fuzzy Tree Parameters

This section introduces threshold profiles and their integrated scalar summaries. A profile records the value of a classical graph parameter at every positive membership threshold, while its integral aggregates that profile into a single real-valued summary.

Throughout this section, whenever a profile defined on $(0, 1]$ is integrated over $[0, 1]$, it is extended to $\alpha = 0$ by assigning the value 0 there. Changing the value at this single point does not affect the Riemann integral. We deliberately do not define a 0-cut graph by the condition $\mu(uv) \geq 0$, since that convention would include zero-membership pairs as edges.

3.1 Alpha-Cut Fuzzy Tree-Width

Alpha-Cut Fuzzy Tree-Width evaluates ordinary tree-width at each positive threshold and integrates the resulting profile. This definition is distinct from the fuzzy-bag construction discussed in [32].

Definition 3.1 (Alpha-Cut Fuzzy Tree-Width). Let $\tilde{G} = (V, \sigma, \mu)$ be a finite fuzzy graph. The *alpha-cut Fuzzy Tree-Width profile* is the function

$$\text{FTW}_{\tilde{G}} : (0, 1] \longrightarrow \mathbb{R}_{\geq 0}$$

defined by

$$\text{FTW}_{\tilde{G}}(\alpha) = \text{tw}(G_{\alpha}).$$

The corresponding *integrated alpha-cut Fuzzy Tree-Width* is

$$\text{IFTW}(\tilde{G}) = \int_0^1 \text{tw}(G_{\alpha}) d\alpha.$$

3.2 Alpha-Cut Fuzzy Tree-Length

Alpha-Cut Fuzzy Tree-Length evaluates tree-length at each threshold level and integrates the profile to summarize metric decomposition complexity across thresholds.

Definition 3.2 (Alpha-Cut Fuzzy Tree-Length). Let $\tilde{G} = (V, \sigma, \mu)$ be a finite fuzzy graph. The *alpha-cut Fuzzy Tree-Length profile* is

$$\text{FTL}_{\tilde{G}}(\alpha) = \text{tl}(G_{\alpha}), \quad \alpha \in (0, 1].$$

The corresponding *integrated alpha-cut Fuzzy Tree-Length* is

$$\text{IFTL}(\tilde{G}) = \int_0^1 \text{tl}(G_{\alpha}) d\alpha.$$

3.3 Alpha-Cut Fuzzy Tree-Depth

Alpha-Cut Fuzzy Tree-Depth evaluates tree-depth at each threshold level and integrates the profile to summarize hierarchical complexity across thresholds.

Definition 3.3 (Alpha-Cut Fuzzy Tree-Depth). Let $\tilde{G} = (V, \sigma, \mu)$ be a finite fuzzy graph. The *alpha-cut Fuzzy Tree-Depth profile* is

$$\text{FTD}_{\tilde{G}}(\alpha) = \text{td}(G_{\alpha}), \quad \alpha \in (0, 1].$$

The corresponding *integrated alpha-cut Fuzzy Tree-Depth* is

$$\text{IFTD}(\tilde{G}) = \int_0^1 \text{td}(G_{\alpha}) d\alpha.$$

3.4 Alpha-Cut Fuzzy Tree-Breadth

Alpha-Cut Fuzzy Tree-Breadth evaluates tree-breadth at each threshold level and integrates the profile to summarize metric covering complexity across thresholds.

Definition 3.4 (Alpha-Cut Fuzzy Tree-Breadth). Let $\tilde{G} = (V, \sigma, \mu)$ be a finite fuzzy graph. The *alpha-cut Fuzzy Tree-Breadth profile* is

$$\text{FTB}_{\tilde{G}}(\alpha) = \text{tb}(G_\alpha), \quad \alpha \in (0, 1].$$

The corresponding *integrated alpha-cut Fuzzy Tree-Breadth* is

$$\text{IFTB}(\tilde{G}) = \int_0^1 \text{tb}(G_\alpha) d\alpha.$$

3.5 Illustrative Example

The following elementary example shows how the integrated parameters can retain threshold information that is invisible in the support graph.

Example 3.5. Let

$$V = \{a, b, c\}, \quad \sigma(a) = \sigma(b) = \sigma(c) = 1,$$

and fix $\varepsilon \in (0, 1)$. Define

$$\mu(ab) = \mu(bc) = 1, \quad \mu(ac) = \varepsilon.$$

Then the support graph is K_3 , whereas

$$G_\alpha = \begin{cases} K_3, & 0 < \alpha \leq \varepsilon, \\ P_3, & \varepsilon < \alpha \leq 1. \end{cases}$$

Using

$$\text{tw}(K_3) = 2, \quad \text{tw}(P_3) = 1, \quad \text{td}(K_3) = 3, \quad \text{td}(P_3) = 2,$$

and

$$\text{tl}(K_3) = \text{tl}(P_3) = 1, \quad \text{tb}(K_3) = \text{tb}(P_3) = 1,$$

we obtain

$$\text{IFTW}(\tilde{G}) = 1 + \varepsilon, \quad \text{IFTD}(\tilde{G}) = 2 + \varepsilon,$$

and

$$\text{IFTL}(\tilde{G}) = \text{IFTB}(\tilde{G}) = 1.$$

Thus, even though the support graph is always K_3 , changing the positive membership value ε changes the integrated Tree-Width and Tree-Depth. Hence these integrated quantities are not determined by the support graph alone.

4. Basic Properties

We establish well-definedness, finite-sum formulas, threshold monotonicity, structural inequalities, and crisp reduction for the alpha-cut parameters.

Theorem 4.1 (Well-Definedness). *For every finite fuzzy graph $\tilde{G}$, the four profile functions*

$$\text{FTW}_{\tilde{G}}, \quad \text{FTL}_{\tilde{G}}, \quad \text{FTD}_{\tilde{G}}, \quad \text{FTB}_{\tilde{G}}$$

are bounded step functions on $(0, 1]$. Consequently,

$$\text{IFTW}(\tilde{G}), \quad \text{IFTL}(\tilde{G}), \quad \text{IFTD}(\tilde{G}), \quad \text{IFTB}(\tilde{G})$$

are finite and well-defined real numbers.

Proof. Let

$$\Lambda = \{\sigma(v) : v \in V\} \cup \left\{ \mu(uv) : uv \in \binom{V}{2} \right\} \cup \{0, 1\}.$$

Because V is finite, Λ is finite. The vertex set V_α and edge set E_α can change only when α crosses a value in Λ . Hence G_α is constant on each open interval between two consecutive distinct values of Λ . Each of the four profile functions is therefore a step function with only finitely many discontinuities.

For every α , the graph G_α is finite. Thus its Tree-Width and Tree-Depth are finite. Its Tree-Length and Tree-Breadth are also finite by the componentwise decomposition observation above. Therefore all four profiles are bounded nonnegative functions with finitely many discontinuities. Under the $\alpha = 0$ convention stated above, they are Riemann integrable on $[0, 1]$, and their integrals are consequently finite and well-defined. $\square$

Theorem 4.2 (Finite-Sum Representation). *Let*

$$\Lambda = \{\sigma(v) : v \in V\} \cup \left\{ \mu(uv) : uv \in \binom{V}{2} \right\} \cup \{0, 1\},$$

and write its distinct elements in increasing order as

$$0 = \lambda_0 < \lambda_1 < \cdots < \lambda_m = 1.$$

For any

$$p \in \{\text{tw}, \text{tl}, \text{td}, \text{tb}\},$$

define

$$I_p(\tilde{G}) = \int_0^1 p(G_\alpha) d\alpha.$$

Then

$$I_p(\tilde{G}) = \sum_{i=1}^m (\lambda_i - \lambda_{i-1}) p(G_{\lambda_i}).$$

In particular,

$$I_{\text{tw}} = \text{IFTW}, \quad I_{\text{tl}} = \text{IFTL}, \quad I_{\text{td}} = \text{IFTD}, \quad I_{\text{tb}} = \text{IFTB}.$$

Proof. For every $i \in \{1, \dots, m\}$ and every $\alpha \in (\lambda_{i-1}, \lambda_i]$, there is no membership value strictly between λ_{i-1} and λ_i . Hence a vertex or edge belongs to G_α exactly when it belongs to G_{λ_i} , and therefore

$$G_\alpha = G_{\lambda_i}.$$

Thus $p(G_\alpha)$ is constant on each interval $(\lambda_{i-1}, \lambda_i]$. Integrating these finitely many constant pieces gives

$$\int_0^1 p(G_\alpha) d\alpha = \sum_{i=1}^m (\lambda_i - \lambda_{i-1}) p(G_{\lambda_i}),$$

as claimed. $\square$

Proposition 4.3 (Threshold Monotonicity for Width and Depth). *Let $\tilde{G}$ be a finite fuzzy graph and let*

$$0 < \alpha \leq \beta \leq 1.$$

Then

$$\text{FTW}_{\tilde{G}}(\beta) \leq \text{FTW}_{\tilde{G}}(\alpha)$$

and

$$\text{FTD}_{\tilde{G}}(\beta) \leq \text{FTD}_{\tilde{G}}(\alpha).$$

Proof. Since $\alpha \leq \beta$, the threshold construction gives

$$G_\beta \subseteq G_\alpha.$$

Tree-Width and Tree-Depth are monotone under taking subgraphs. Hence

$$\text{tw}(G_\beta) \leq \text{tw}(G_\alpha)$$

and

$$\text{td}(G_\beta) \leq \text{td}(G_\alpha),$$

which proves the stated inequalities. $\square$

Theorem 4.4 (Structural Inequalities). *For every finite fuzzy graph $\tilde{G}$ and every $\alpha \in (0, 1]$,*

$$\text{FTB}_{\tilde{G}}(\alpha) \leq \text{FTL}_{\tilde{G}}(\alpha) \leq 2 \text{FTB}_{\tilde{G}}(\alpha)$$

and

$$\text{FTW}_{\tilde{G}}(\alpha) \leq \text{FTD}_{\tilde{G}}(\alpha).$$

Consequently,

$$\text{IFTB}(\tilde{G}) \leq \text{IFTL}(\tilde{G}) \leq 2 \text{IFTB}(\tilde{G})$$

and

$$\text{IFTW}(\tilde{G}) \leq \text{IFTD}(\tilde{G}).$$

Proof. Let H be a finite graph. For every nonempty vertex set X having finite diameter, choosing any vertex of X as a center gives

$$\text{rad}_H(X) \leq \text{diam}_H(X).$$

Conversely, if X has finite radius and c is a center attaining $\text{rad}_H(X)$, then the triangle inequality gives

$$d_H(u, v) \leq d_H(u, c) + d_H(c, v) \leq 2 \text{rad}_H(X)$$

for all $u, v \in X$. Hence

$$\text{diam}_H(X) \leq 2 \text{rad}_H(X).$$

A componentwise tree-decomposition shows that both $\text{tl}(H)$ and $\text{tb}(H)$ are finite. Let $\mathcal{T}_\ell$ attain $\text{tl}(H)$. Every bag of $\mathcal{T}_\ell$ has finite diameter, so

$$\text{breadth}_H(\mathcal{T}_\ell) \leq \text{len}_H(\mathcal{T}_\ell) = \text{tl}(H).$$

Therefore

$$\text{tb}(H) \leq \text{tl}(H).$$

Likewise, let $\mathcal{T}_b$ attain $\text{tb}(H)$. Every bag of $\mathcal{T}_b$ has finite radius, and hence

$$\text{len}_H(\mathcal{T}_b) \leq 2 \text{breadth}_H(\mathcal{T}_b) = 2 \text{tb}(H).$$

Thus

$$\text{tl}(H) \leq 2 \text{tb}(H).$$

For the width–depth relation, let H be nonempty and let F be a rooted forest of height h such that

$$H \subseteq \text{clos}(F).$$

For each $x \in V(F)$, let X_x consist of x together with all ancestors of x in F . After joining the components of F by arbitrary edges between their roots, these bags form a tree-decomposition of $\text{clos}(F)$: every ancestor–descendant edge is contained in the bag indexed by its deeper endpoint, and the bags containing a fixed vertex form the subtree induced by its descendants. Since $|X_x| \leq h$ for every x , this decomposition has width at most $h - 1$. By subgraph monotonicity,

$$\text{tw}(H) \leq h - 1.$$

Minimizing over all rooted forests witnessing $\text{td}(H)$ yields

$$\text{tw}(H) \leq \text{td}(H) - 1 \leq \text{td}(H).$$

For the empty graph, both parameters are 0 by convention. Applying these inequalities to $H = G_\alpha$ proves the pointwise profile inequalities. Integrating them over $[0, 1]$ gives the corresponding inequalities for the integrated parameters. $\square$

Theorem 4.5 (Crisp Reduction). *Let $G = (V, E)$ be a finite simple graph and let $\tilde{G}$ be its crisp fuzzy representation defined by*

$$\sigma(v) = 1 \quad (v \in V)$$

and

$$\mu(uv) = \begin{cases} 1, & uv \in E, \\ 0, & uv \notin E. \end{cases}$$

Then, for every $\alpha \in (0, 1]$,

$$G_\alpha = G.$$

Consequently,

$$\text{IFTW}(\tilde{G}) = \text{tw}(G),$$

$$\text{IFTL}(\tilde{G}) = \text{tl}(G),$$

$$\text{IFTD}(\tilde{G}) = \text{td}(G),$$

and

$$\text{IFTB}(\tilde{G}) = \text{tb}(G).$$

Proof. For every $\alpha \in (0, 1]$, all vertices satisfy $\sigma(v) \geq \alpha$, while an unordered pair uv satisfies $\mu(uv) \geq \alpha$ if and only if $uv \in E$. Hence $G_\alpha = G$ for every $\alpha \in (0, 1]$. Each profile is therefore constant on $(0, 1]$ and equal to the corresponding classical graph parameter. Integrating a constant function over an interval of length one yields the four stated identities. $\square$

Remark 4.6. The alpha-cut construction and the fuzzy-bag construction of Fuzzy Tree-Width encode different information. The former studies the complete family of crisp threshold graphs $\{G_\alpha\}_{\alpha \in (0,1]}$, whereas the latter places membership values directly inside the bags of a fuzzy tree-decomposition. Accordingly, IFTW should be interpreted as an integrated alpha-cut parameter rather than as a replacement for previously defined fuzzy-bag Tree-Width.

5. Connections with Fuzzy Decision Trees

A Fuzzy Decision Tree uses fuzzy branch-membership degrees for classification or inference [36]. Its operational depth, measured by the number of internal test nodes on a longest root-to-terminal path, differs from the graph-theoretic Tree-Depth of its underlying undirected tree.

We first construct a fuzzy graph on test-node occurrences for which the alpha-cut parameters admit exact expressions.

Definition 5.1 (Normalized Product-Routing Fuzzy Decision Tree). Let $\mathcal{D}$ be a finite rooted decision tree with nonempty set N of internal test nodes, root $r \in N$, and terminal output leaves.

Let $\mathcal{X}$ be a nonempty finite input set, with weights $\pi_x \geq 0$ satisfying

$$\sum_{x \in \mathcal{X}} \pi_x = 1.$$

For every arc $t \rightarrow c$ leaving an internal test node, suppose that a branch-membership function

$$q_{t,c} : \mathcal{X} \longrightarrow [0, 1]$$

is specified and satisfies

$$\sum_{c: t \rightarrow c} q_{t,c}(x) = 1 \quad (t \in N, x \in \mathcal{X}).$$

For $t \in N$, set

$$w_t(x) = \prod_{(u \rightarrow v) \in P(r,t)} q_{u,v}(x),$$

and

$$a_t = \sum_{x \in \mathcal{X}} \pi_x w_t(x),$$

where $P(r, t)$ is the set of arcs on the unique directed path from r to t , and the empty product equals 1.

We call a_t the *average activation degree* of test node t .

The rooted tree induced on N by $\mathcal{D}$ is denoted by F , and $d = h(F)$ counts internal test nodes, rather than arcs, on a longest root-to-test-node path.

Definition 5.2 (Decision-Tree-Induced Fuzzy Graph). With the notation above, define

$$\tilde{H}_{\mathcal{D}} = (N, \sigma_{\mathcal{D}}, \mu_{\mathcal{D}}), \quad \sigma_{\mathcal{D}}(t) = a_t,$$

and, for distinct $s, t \in N$,

$$\mu_{\mathcal{D}}(st) = \begin{cases} a_t, & \text{if } s \text{ is an ancestor of } t \text{ in } F, \\ a_s, & \text{if } t \text{ is an ancestor of } s \text{ in } F, \\ 0, & \text{otherwise.} \end{cases}$$

Here, vertices represent *test-node occurrences*; different nodes may test the same original input feature.

Theorem 5.3 (Exact Width–Depth Relations for a Fuzzy Decision Tree). *The graph $\tilde{H}_{\mathcal{D}}$ is a well-defined finite fuzzy graph.*

Let

$$N_j = \{t \in N : \text{depth}_F(t) = j\},$$

and

$$m_j = \max_{t \in N_j} a_t \quad (1 \leq j \leq d),$$

where the root has depth 1.

Then

$$\text{IFTD}(\tilde{H}_{\mathcal{D}}) = \sum_{j=1}^d m_j,$$

and

$$\text{IFTW}(\tilde{H}_{\mathcal{D}}) = \sum_{j=2}^d m_j.$$

In particular,

$$\text{IFTW}(\tilde{H}_{\mathcal{D}}) = \text{IFTD}(\tilde{H}_{\mathcal{D}}) - 1.$$

Proof. The root has $a_r = 1$.

If s is an ancestor of t , then $w_t(x) \leq w_s(x)$ for every input x because each branch grade lies in $[0, 1]$.

Hence $a_t \leq a_s$, and the membership compatibility condition

$$\mu_{\mathcal{D}}(st) \leq \min\{a_s, a_t\}$$

holds.

Fix $\alpha \in (0, 1]$ and let

$$N_{\alpha} = \{t \in N : a_t \geq \alpha\}.$$

The set N_{α} contains the root and is closed under taking ancestors.

Consequently,

$$F_{\alpha} = F[N_{\alpha}]$$

is a rooted tree, and the definition of $\mu_{\mathcal{D}}$ gives

$$(\tilde{H}_{\mathcal{D}})_{\alpha} = \text{clos}(F_{\alpha}).$$

Write $h_{\alpha} = h(F_{\alpha})$.

A longest ancestor chain is a clique of size h_{α} in this closure, so

$$\text{td}((\tilde{H}_{\mathcal{D}})_{\alpha}) \geq h_{\alpha},$$

and

$$\text{tw}((\tilde{H}_{\mathcal{D}})_{\alpha}) \geq h_{\alpha} - 1.$$

The forest F_{α} witnesses the reverse Tree-Depth inequality.

Moreover, the bags consisting of each node together with its ancestors form a tree-decomposition of width at most $h_{\alpha} - 1$.

Thus

$$\text{td}((\tilde{H}_{\mathcal{D}})_{\alpha}) = h_{\alpha},$$

and

$$\text{tw}((\tilde{H}_{\mathcal{D}})_{\alpha}) = h_{\alpha} - 1.$$

By ancestor monotonicity, a node at depth j survives the cut if and only if its activation is at least α .

Hence

$$h_{\alpha} = \sum_{j=1}^d \mathbf{1}_{\{m_j \geq \alpha\}}.$$

Integrating over $\alpha \in [0, 1]$ yields

$$\text{IFTD}(\tilde{H}_{\mathcal{D}}) = \sum_{j=1}^d m_j.$$

Since $m_1 = a_r = 1$, the identity

$$\text{IFTW} = \text{IFTD} - 1$$

gives the remaining formula. □

Corollary 5.4 (Soft Path-Length Bounds). *For the soft path-length functional*

$$C_{\text{soft}}(\mathcal{D}) = \sum_{t \in N} a_t,$$

one has

$$1 \leq \text{IFTD}(\tilde{H}_{\mathcal{D}}) \leq C_{\text{soft}}(\mathcal{D}) \leq d.$$

Proof. The first inequality follows from $m_1 = 1$.

For every level j ,

$$m_j \leq \sum_{t \in N_j} a_t,$$

so summing gives the second inequality.

The normalized branch grades ensure that the total activation

$$\sum_{t \in N_j} w_t(x)$$

of test nodes at each level is at most 1.

Averaging over x gives

$$\sum_{t \in N_j} a_t \leq 1.$$

Summing over at most d levels proves

$$C_{\text{soft}}(\mathcal{D}) \leq d.$$

□

Example 5.5 (A Two-Level Fuzzy Decision Tree). Suppose that the root test node r has two test-node children b, c , with constant routing grades 0.7 and 0.3, respectively, and that b and c lead directly to terminal outputs.

Then

$$a_r = 1, \quad a_b = 0.7, \quad a_c = 0.3,$$

and

$$\mu_{\mathcal{D}}(rb) = 0.7, \quad \mu_{\mathcal{D}}(rc) = 0.3, \quad \mu_{\mathcal{D}}(bc) = 0.$$

Here $m_1 = 1$, $m_2 = 0.7$, and $d = 2$.

Thus

$$\text{IFTD}(\tilde{H}_{\mathcal{D}}) = 1.7,$$

and

$$\text{IFTW}(\tilde{H}_{\mathcal{D}}) = 0.7,$$

while

$$C_{\text{soft}}(\mathcal{D}) = 2.$$

Remark 5.6. The preceding construction applies to normalized product-routing Fuzzy Decision Trees; other fuzzy routing conventions need not satisfy the soft path-length bound.

The graph is defined on test-node *occurrences*, not on distinct features.

Consequently, these identities do not provide a bound on predictive accuracy, learning complexity, or the evaluation cost of implementations that activate several branches simultaneously.

Definition 5.7 (Feature-Interaction-Compatible Forest). Let $\tilde{G} = (V, \sigma, \mu)$ be a finite fuzzy graph and let

$$G^+ = (V^+, E^+)$$

be its support graph.

A rooted forest F on V^+ is called *feature-interaction-compatible* if

$$G^+ \subseteq \text{clos}(F).$$

For $\alpha \in (0, 1]$, let $F|_{V_\alpha}$ denote the rooted forest on V_α obtained by linking each retained vertex to its nearest retained ancestor, if one exists.

Set $h(\emptyset) = 0$.

Theorem 5.8 (Bounds for Compatible Feature Hierarchies). *Let $\tilde{G}$ be a finite fuzzy graph, and let F be a rooted forest satisfying $G^+ \subseteq \text{clos}(F)$. Put*

$$s_{\max} = \max(\{\sigma(v) : v \in V\} \cup \{0\}).$$

For every $\alpha \in (0, 1]$,

$$\text{FTD}_{\tilde{G}}(\alpha) \leq h(F|_{V_\alpha}),$$

and

$$\text{FTW}_{\tilde{G}}(\alpha) \leq h(F|_{V_\alpha}) - \mathbf{1}_{\{V_\alpha \neq \emptyset\}}.$$

Consequently,

$$\text{IFTD}(\tilde{G}) \leq \int_0^1 h(F|_{V_\alpha}) d\alpha \leq s_{\max} h(F),$$

and

$$\text{IFTW}(\tilde{G}) \leq \int_0^1 h(F|_{V_\alpha}) d\alpha - s_{\max}.$$

Proof. Each edge of G_α is an edge of G^+ , so its endpoints are ancestor-comparable in F .

Deleting vertices outside V_α and reconnecting retained vertices to their nearest retained ancestors preserves this comparability.

Hence

$$G_\alpha \subseteq \text{clos}(F|_{V_\alpha}).$$

The definition of Tree-Depth gives the first pointwise inequality.

The canonical ancestor-bag decomposition gives the second, with the empty-graph convention accounting for the indicator.

Furthermore, $F|_{V_\alpha}$ has height at most $h(F)$ and is empty when $\alpha > s_{\max}$.

Integrating the pointwise inequalities and using

$$\int_0^1 \mathbf{1}_{\{V_\alpha \neq \emptyset\}} d\alpha = s_{\max}$$

proves the assertions. □

Remark 5.9. If the forest F is interpreted as the hierarchy of feature tests in a decision model, its vertices must represent distinct features, or else the model must explicitly distinguish repeated test occurrences.

Compatibility is a structural restriction and does not imply that a given target classifier can be represented by that hierarchy without additional assumptions.

Theorem 5.10 (Integrated Width–Depth Gap). *For every finite fuzzy graph $\tilde{G} = (V, \sigma, \mu)$, let*

$$s_{\max} = \max(\{\sigma(v) : v \in V\} \cup \{0\}).$$

Then

$$\text{IFTW}(\tilde{G}) \leq \text{IFTD}(\tilde{G}) - s_{\max}.$$

Proof. For every nonempty finite graph H ,

$$\text{tw}(H) \leq \text{td}(H) - 1;$$

for the empty graph both parameters are 0.

Applying this to G_α yields

$$\text{tw}(G_\alpha) \leq \text{td}(G_\alpha) - \mathbf{1}_{\{V_\alpha \neq \emptyset\}}.$$

The set V_α is nonempty exactly when

$$0 < \alpha \leq s_{\max}.$$

Integrating proves the claim. □

Remark 5.11. There is no converse bound on Tree-Depth depending only on Tree-Width.

For example,

$$\text{tw}(P_n) = 1, \quad \text{td}(P_n) = \lceil \log_2(n+1) \rceil \quad (n \geq 2).$$

By crisp reduction, the same distinction applies to the integrated alpha-cut parameters.

Thus a small Fuzzy Tree-Width alone does not guarantee a uniformly shallow feature-testing hierarchy.

6. Conclusion

We developed an alpha-cut framework for Tree-Width, Tree-Length, Tree-Depth, and Tree-Breadth on finite undirected fuzzy graphs. By applying each classical parameter to the positive α -cut graphs, we obtained threshold profiles and integrated summaries that retain information about membership levels beyond that contained in the support graph. All four profiles are bounded step functions, and their integrals admit exact finite-sum representations. We also established crisp reduction and proved that Tree-Width and Tree-Depth are nonincreasing as the threshold increases. For the metric parameters, we obtained

$$\text{IFTB}(\tilde{G}) \leq \text{IFTL}(\tilde{G}) \leq 2 \text{IFTB}(\tilde{G}).$$

Moreover, writing

$$s_{\max} = \max(\{\sigma(v) : v \in V\} \cup \{0\}),$$

we strengthened the width–depth relation to

$$\text{IFTW}(\tilde{G}) \leq \text{IFTD}(\tilde{G}) - s_{\max}.$$

We further connected these graph parameters with a specified class of Fuzzy Decision Trees. For a normalized product-routing tree, the fuzzy graph induced by test-node activations and ancestor relationships satisfies

$$\text{IFTD}(\tilde{H}_{\mathcal{D}}) = \sum_{j=1}^d m_j, \quad \text{IFTW}(\tilde{H}_{\mathcal{D}}) = \sum_{j=2}^d m_j,$$

where m_j is the maximum average activation at test-node level j . In particular,

$$\text{IFTW}(\tilde{H}_{\mathcal{D}}) = \text{IFTD}(\tilde{H}_{\mathcal{D}}) - 1.$$

We also bounded integrated depth by the soft path-length functional and obtained structural bounds from feature-interaction-compatible rooted forests. These statements concern the specific graph constructions and do not identify graph-theoretic Tree-Depth with a decision tree's operational depth in general.

The proposed quantities are integrated alpha-cut parameters, not replacements for existing fuzzy-bag definitions. The present work is theoretical; no predictive-performance guarantees or general-purpose learning algorithms are established. Further research may investigate algorithms for computing the integrated parameters, additional properties of fuzzy decision-tree representations, and extensions to Neutrosophic Graphs [13, 42], Plithogenic Graphs [43], HyperGraphs [44, 45], and SuperHyperGraphs [46].

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Conflicts of Interest

The authors declare that there are no conflicts of interest related to this research or its publication.

References

- [1] Zadeh, L. A. (1965). Fuzzy sets. *Information and control*, 8(3), 338–353. [https://doi.org/10.1016/S0019-9958\(65\)90241-X](https://doi.org/10.1016/S0019-9958(65)90241-X)
- [2] Das, S., & Kar, S. (2013). Intuitionistic multi fuzzy soft set and its application in decision making. *Pattern Recognition and Machine Intelligence: 5th International Conference, PReMI 2013, Kolkata, India, December 10-14, 2013. Proceedings 5*, 587–592. https://doi.org/10.1007/978-3-642-45062-4_82
- [3] Atanassov, K. T., & Gargov, G. (2017). *Intuitionistic fuzzy logics*. Springer.
- [4] Wang, H., Smarandache, F., Zhang, Y., & Sunderraman, R. (2010). *Single valued neutrosophic sets*. Infinite study. <https://fs.unm.edu/neut/SingleValuedNeutrosophicSets.pdf>
- [5] Smarandache, F. (1999). A unifying field in logics: Neutrosophic logic. In *Philosophy* (pp. 1–141). American Research Press. <https://arxiv.org/pdf/math/0101228>
- [6] Nordo, G., Smarandache, F., et al. (2027). Plithogenic topological spaces. *NEUTROSOPHIC SETS AND SYSTEMS*, 103, 17–47. <https://doi.org/10.5281/zenodo.22821085>
- [7] Angel, N., Pandiammal, P., & Martin, N. (2026). Plithogenic cognitive maps with extended plithogenic representations & applications. *New Trends in Neutrosophic Theories and Applications, Volume V*, 126. <https://doi.org/10.5281/zenodo.20426406>
- [8] Rosenfeld, A. (1975). Fuzzy graphs. In *Fuzzy sets and their applications to cognitive and decision processes* (pp. 77–95). Elsevier.

- [9] Al-Hawary, T. (2011). Complete fuzzy graphs. *International Journal of Mathematical Combinatorics*, 4, 26.
- [10] Mordeson, J. N., & Nair, P. S. (2012). *Fuzzy graphs and fuzzy hypergraphs* (Vol. 46). Physica. <https://doi.org/10.1007/978-3-7908-1854-3>
- [11] Broumi, S., Bakali, A., Talea, M., Smarandache, F., & Verma, R. (2022). Computing minimum spanning tree in interval valued bipolar neutrosophic environment. *Collected Papers. Volume VII: On Neutrosophic Theory and Applications*, 7(5), 143.
- [12] Broumi, S., Bakali, A., & Bannasse, A. (2018). Neutrosophic sets: An overview. *Infinite Study*. https://www.researchgate.net/publication/328019074_Neutrosophic_Sets_An_Overview
- [13] Broumi, S., Talea, M., Bakali, A., & Smarandache, F. (2016a). Interval valued neutrosophic graphs. *Critical Review, XII, 2016*, 5–33. https://ie.utcluj.ro/files/acta/2017/Number1-2/paper__16.pdf
- [14] Broumi, S., Talea, M., Bakali, A., Smarandache, F., & Kumar, P. K. (2017). Shortest path problem on single valued neutrosophic graphs. *2017 international symposium on networks, computers and communications (ISNCC)*, 1–6. <https://doi.org/10.1109/ISNCC.2017.8071993>
- [15] Zhao, Z., & Zhang, X. (2026). Frf-fhg: Constructing fuzzy hypergraphs based on frf. *Journal of Intelligence and Knowledge Engineering (ISSN: 2959-0620)*, 4(1), 209. <https://doi.org/10.62517/jike.202604130>
- [16] Datta, R., Baitalik, S., Das Choudhury, S., Chattopadhyay, A. K., & Das, A. K. (2026). Fuzzy hypergraph feature association map for high-dimensional feature selection in agriculture and remote sensing. *International Journal of Fuzzy Systems*, 1–14. <https://doi.org/10.1007/s40815-026-02267-2>
- [17] Fujita, T. (2026). Tripolar fuzzy hypergraph and tripolar fuzzy superhypergraph. *Journal of Mathematics and Informatics*, 31, 5–18. <https://doi.org/10.22457/jmi.v31a02275>
- [18] Robertson, N., & Seymour, P. D. (1984). Graph minors. iii. planar tree-width. *Journal of Combinatorial Theory, Series B*, 36(1), 49–64. [https://doi.org/10.1016/0095-8956\(84\)90013-3](https://doi.org/10.1016/0095-8956(84)90013-3)
- [19] Bodlaender, H. L. (1997). Treewidth: Algorithmic techniques and results. *International Symposium on Mathematical Foundations of Computer Science*, 19–36. <https://doi.org/10.1007/BFb0029946>
- [20] Bodlaender, H. L., Grigoriev, A., & Koster, A. M. (2005). Treewidth lower bounds with brambles. *Algorithms–ESA 2005: 13th Annual European Symposium, Palma de Mallorca, Spain, October 3-6, 2005. Proceedings 13*, 391–402.
- [21] Robertson, N., & Seymour, P. D. (1983). Graph minors. i. excluding a forest. *Journal of Combinatorial Theory, Series B*, 35(1), 39–61. [https://doi.org/10.1016/0095-8956\(83\)90079-5](https://doi.org/10.1016/0095-8956(83)90079-5)
- [22] Kaplan, H., & Shamir, R. (1996). Pathwidth, bandwidth, and completion problems to proper interval graphs with small cliques. *SIAM Journal on Computing*, 25(3), 540–561. <https://doi.org/10.1137/S0097539793258143>
- [23] Dourisboure, Y., Dragan, F. F., Gavaille, C., & Yan, C. (2007). Spanners for bounded tree-length graphs. *Theoretical Computer Science*, 383(1), 34–44. <https://doi.org/10.1016/j.tcs.2007.03.058>

- [24] Leitert, A., & Dragan, F. F. (2016). On strong tree-breadth. *International Conference on Combinatorial Optimization and Applications*, 62–76. https://doi.org/10.1007/978-3-319-48749-6_5
- [25] Nešetřil, J., & De Mendez, P. O. (2006). Tree-depth, subgraph coloring and homomorphism bounds. *European Journal of Combinatorics*, 27(6), 1022–1041. <https://doi.org/10.1016/j.ejc.2005.01.010>
- [26] Gottlob, G., Leone, N., & Scarcello, F. (1999). Hypertree decompositions and tractable queries. *Proceedings of the eighteenth ACM SIGMOD-SIGACT-SIGART symposium on Principles of database systems*, 21–32.
- [27] Gottlob, G., Leone, N., & Scarcello, F. (2001). Hypertree decompositions: A survey. *Mathematical Foundations of Computer Science 2001: 26th International Symposium, MFCS 2001 Mariánské Lázně, Czech Republic, August 27–31, 2001 Proceedings* 26, 37–57. https://doi.org/10.1007/3-540-44683-4_5
- [28] Robertson, N., & Seymour, P. D. (1991). Graph minors. x. obstructions to tree-decomposition. *Journal of Combinatorial Theory, Series B*, 52(2), 153–190. [https://doi.org/10.1016/0095-8956\(91\)90061-N](https://doi.org/10.1016/0095-8956(91)90061-N)
- [29] Oum, S.-i. (2008). Rank-width is less than or equal to branch-width. *Journal of Graph Theory*, 57(3), 239–244. <https://doi.org/10.1002/jgt.20280>
- [30] Munaro, A., & Yang, S. (2023). On algorithmic applications of sim-width and mim-width of (h_1, h_2) -free graphs. *Theoretical Computer Science*, 955, 113825. <https://doi.org/10.1016/j.tcs.2023.113825>
- [31] Fujita, T., & Smarandache, F. (2026). *Fuzzy, neutrosophic, and uncertain graph theory: Properties and applications*. Infinite Study.
- [32] Fujita, T., Batiha, I. M., Gulistan, M., Batiha, B., Al Smadi, E. L., & Jebril, I. (2026). Tree decompositions in fuzzy graphs: Foundations and structural insights. *Statistics, Optimization & Information Computing*, 16(1), 312–326. <https://doi.org/10.19139/soic-2310-5070-3117>
- [33] Kotsiantis, S. B. (2013). Decision trees: A recent overview. *Artificial Intelligence Review*, 39, 261–283. <https://doi.org/10.1007/s10462-011-9272-4>
- [34] Murthy, S. K. (1998). Automatic construction of decision trees from data: A multi-disciplinary survey. *Data mining and knowledge discovery*, 2(4), 345–389. <https://doi.org/10.1023/A:1009744630224>
- [35] Olaru, C., & Wehenkel, L. (2003). A complete fuzzy decision tree technique. *Fuzzy sets and systems*, 138(2), 221–254. [https://doi.org/10.1016/S0165-0114\(03\)00089-7](https://doi.org/10.1016/S0165-0114(03)00089-7)
- [36] Janikow, C. Z. (1998). Fuzzy decision trees: Issues and methods. *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)*, 28(1), 1–14. <https://doi.org/10.1109/3477.658573>
- [37] Pang, B. (2025). Neutrosophic information gain for decision tree construction: Application to teaching performance of tennis instructors in sports universities. *Neutrosophic Sets and Systems*, 91(1), 9. https://digitalrepository.unm.edu/cgi/viewcontent.cgi?article=3715&context=nss_journal

- [38] Olgun, N., & Hatip, A. (2020). The effect of the neutrosophic logic on the decision tree. *Quadruple Neutrosophic Theory and Applications, Pons Editions Brussels, Belgium, EU*, 17, 238–253. https://books.google.co.jp/books?id=GlfIEAAAQBAJ&printsec=frontcover&source=gbg_atb&redir_esc=y#v=onepage&q&f=false
- [39] Kloks, T. (1994). *Treewidth: Computations and approximations*. Springer. <https://doi.org/https://doi.org/10.1007/BFb0045375>
- [40] Bonnet, É. (2024). Treewidth inapproximability and tight eth lower bound. *arXiv preprint arXiv:2406.11628*.
- [41] Gottlob, G., Greco, G., Scarcello, F., et al. (2014). Treewidth and hypertree width. *Tractability: Practical Approaches to Hard Problems*, 1, 20. <https://www.semanticscholar.org/paper/Treewidth-and-Hypertree-Width-Gottlob-Greco/959ee121656ff4f0a3449f7bac4521b61ea9c995>
- [42] Broumi, S., Talea, M., Bakali, A., & Smarandache, F. (2016b). Single valued neutrosophic graphs. *Journal of New theory*, 10, 86–101. <https://doi.org/10.5281/zenodo.50940>
- [43] Smarandache, F. (2020). *Extension of hypergraph to n-superhypergraph and to plithogenic n-superhypergraph, and extension of hyperalgebra to n-ary (classical-/neutro-/anti-) hyperalgebra*. Infinite Study. <https://fs.unm.edu/nss8/index.php/111/article/view/198>
- [44] Bretto, A. (2013). Hypergraph theory. *An introduction. Mathematical Engineering*. Cham: Springer, 1. <https://doi.org/https://doi.org/10.1007/978-3-319-00080-0>
- [45] Feng, Y., You, H., Zhang, Z., Ji, R., & Gao, Y. (2019). Hypergraph neural networks. *Proceedings of the AAAI conference on artificial intelligence*. <https://arxiv.org/abs/1809.09401>
- [46] Ghods, M., Rostami, Z., & Smarandache, F. (2022). Introduction to neutrosophic restricted superhypergraphs and neutrosophic restricted superhypertrees and several of their properties. *Neutrosophic Sets and Systems*, 50, 480–487. https://digitalrepository.unm.edu/nss_journal/vol50/iss1/28/